How the Einstein Equivalence Principle appears in non-relativistic field theories

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We show that the Einstein Equivalence Principle holds in non-relativistic Lagrangian field theories with translational symmetry in the preferred coordinates, if one identifies the stress–energy–momentum tensor with the gravitational field and all other fields with matter fields.

I. INTRODUCTION

In [1], a metric theory of gravity has been proposed where the gravitational field is not fundamental, but emergent. It is defined by fundamental condensed matter variables on a classical Newtonian background by:

$$g^{00}\sqrt{-g} = \rho \tag{1}$$
$$g^{0i}\sqrt{-g} = \rho v^i \tag{2}$$
$$g^{ij}\sqrt{-g} = \rho v^i v^j - \sigma^{ij}. \tag{3}$$

The fundamental theory depends on preferred coordinates. Therefore the Strong Equivalence Principle cannot hold. Nonetheless, in this theory the Einstein Equivalence Principle (EEP) holds exactly. Moreover, it has been derived. Thus, only the equations of the gravitational field depends on the preferred coordinates.

The aim of this paper is to clarify which properties of a general non-relativistic theory are essential for deriving the EEP.

We find that the existence of a Lagrange formalism together with translational symmetry in the preferred coordinates is already sufficient to derive an analogon of the EEP.

The trick is to identify the gravitational field with the stress–energy–momentum tensor of the theory. All other fields will play the role of the matter fields. The equation for the preferred coordinates – the Noether conservation laws – depend only on the stress–energy–momentum tensor. So, by construction, they will depend only on the "gravitational field", and not depend on any "matter fields". Therefore, following the action equals reaction symmetry, the equations for those "matter fields" do not depend on the preferred coordinates too. This proves the EEP.

II. THE COVARIANT FORM OF A LAGRANGIAN OF A THEORY WITH PREFERRED COORDINATES

We assume a non-relativistic field theory with a preferred frame on $\mathbb{R}^3 \times \mathbb{R}$. We will denote general coordinates with x^μ resp. simply x, and use fractal letters for the preferred coordinates $\mathfrak{t} = \mathfrak{x}^0$ for absolute time and $\mathfrak{x}^i$ for the Cartesian spatial coordinates.

Given that the Lagrangian as well as the equations of the theory depend on the preferred coordinates, it would be useful if we could consider the corresponding functional derivatives $\frac{\delta}{\delta \mathfrak{x}^\alpha}$. This presumes that minor modifications of the preferred coordinates, $\mathfrak{x}^\alpha \to \mathfrak{x}^\alpha + \delta\mathfrak{x}^\alpha$, have a well-defined meaning. So we restrict ourself to Lagrangians where we know how the Lagrangian changes if we vary the preferred coordinates. A large class of such Lagrangians is defined by the requirement that all dependencies of the preferred coordinates are given in explicit form as a dependence on the four scalar functions $\mathfrak{x}^\alpha(x)$ and their partial derivatives $\mathfrak{x}^\alpha_{,m}(x)$. Except for using the explicit dependence on the preferred coordinates $\mathfrak{x}^\alpha(x)$ as if they were scalar fields, the Lagrangian has to be covariant.

So, instead of varying a component of a four-vector $a^\alpha(\mathfrak{x})$ given in the preferred coordinates, we vary the equivalent covariant expression $a^\mu(x)\mathfrak{x}^\alpha_{,\mu}(x)$ given in general coordinates. This expression is covariant if one considers the four functions $\mathfrak{x}^\alpha(x)$ as four independent scalar functions which don't have to be transformed like tensor indices for coordinate transformations $x' = x'(x)$. We will use the first greek indices $\alpha, \beta, \ldots$ (instead of the indices $\mu, \nu, \ldots$ used for tensor indices) to denote indices which enumerate the preferred coordinates.

Together with the preferred coordinates, we have also other fields $u^p(x)$, some collection of arbitrary components of tensor fields. The resulting Lagrangian is

$$S = \int \mathcal{L}(u^p(x), u^p_{,\mu}(x), \ldots, \mathfrak{x}^\alpha(x), \mathfrak{x}^\alpha_{,\mu}(x), \ldots) d^4x \tag{4}$$

A. Is the condition of formal covariance restrictive?

Following Kretschmann's objection [2] all classical physical theories can be presented in a covariant form. Given that the way to present a non-relativistic theory in a formally covariant form is a special one, it makes sense to look if this restricts the class of non-relativistic theories which can be made fit into this scheme.

First, all fields which are components of four-dimensional tensor fields can be described in such a way. Indeed, for an upper index we have already given the rule:

$$T^{\ldots\alpha}_{\ldots} \to T^{\ldots\mu}_{\ldots}\mathfrak{x}^\alpha_{,\mu}.$$

For lower indices, we can do the same with the inverse matrix:

$$T^{\ldots}_{\ldots\alpha} \to T^{\ldots}_{\ldots\mu}\frac{\partial x^\mu}{\partial \mathfrak{x}^\alpha}.$$

But the inverse matrix $\frac{\partial x^\mu}{\partial \mathfrak{x}^\alpha}$ is simply a polynomial of the $\mathfrak{x}^\alpha_{,\mu} = \frac{\partial \mathfrak{x}^\alpha}{\partial x^\mu}$.

What could be done if the objects of the theory are purely spatial, three-dimensional tensor fields, or purely temporal? There may be many different ways to do this, but there is, fortunately, no necessity in this context to identify the best way to do this. All one needs is that there exists at least one possibility. This could be defined by replacing spatial indices with four-dimensional ones, and adding the condition that the undefined temporal components are zero in the preferred frame.

So, the condition to present the theory in a formally covariant form so that all the dependencies on the preferred coordinates are given as explicit dependencies on four scalar functions $\mathfrak{x}^\alpha(x)$ and their partial derivatives does not seem to restrict the generality of the approach.

III. NOETHER'S THEOREM

Let's now assume that we have, in the preferred coordinates, translational symmetry. That means that the Lagrangian does not depend on constants added to the preferred coordinates $\mathfrak{x}^\alpha(x) \to \mathfrak{x}^\alpha(x)+c^\alpha$. But if we have such a symmetry, then the first term in the Euler-Lagrange equations $\frac{\partial S}{\partial \mathfrak{x}^\alpha}$ will be zero, the Lagrangian depends only on the partial derivatives of the preferred coodinates, not on their values themselves. As a consequence, the Euler-Lagrange equations automatically become conservation laws:

$$\frac{\delta S}{\delta \mathfrak{x}^\alpha} = \frac{\partial S}{\partial \mathfrak{x}^\alpha} - \partial_\mu \left(\frac{\partial S}{\partial \mathfrak{x}^\alpha_{,\mu}} - \partial_\nu(\dots) \right) = \partial_\mu T^\mu_\alpha = 0.$$

with

$$T^\mu_\alpha = -\frac{\partial S}{\partial \mathfrak{x}^\alpha_{,\mu}} - \partial_\nu(\dots)$$

So, the assumption that the Euler-Lagrange equations for the preferred coordinates define the Noether conservation laws related with translational symmetry is not at all nontrivial, but a variant of Noether's theorem. It holds for every theory with preferred coordinates and translational symmetry for the translations defined by these preferred coordinates.

IV. THE GRAVITATIONAL FIELD AS THE STRESS-ENERGY-MOMENTUM TENSOR OF THE THEORY

In [1] the gravitational field is defined simply as the stress–energy–momentum tensor of the theory, by the following formula:

$$g^{\mu\nu}\sqrt{-g} = T^{\mu\nu}.$$

What can we do here in the most general case? We can simply use the stress–energy–momentum tensor as what defines the gravitational field. In the most general case, the stress–energy–momentum tensor may be an asymmetric tensor with 16 independent field components. In this case, we would simply use this tensor field as the gravitational field. In other cases, there are less independent components. So, in the case considered in [1], there is a constant metric $\gamma^{\alpha\beta}$ so that the tensor $T^{\mu\alpha} = \gamma^{\alpha\beta} T^\mu_\beta$ is symmetric. So, the stress–energy–momentum tensor has only ten independent components, which define the metric tensor field as the gravitational field. We can, as well, have cases with even less degrees of freedom. So, instead of a full stress tensor $\sigma^{ij}(x)$ with six independent components as usual for solids there may be only a scalar pressure $\sigma^{ij}(x) = p(x)\delta^{ij}$ as we have in liquids or gases.

In all these cases we have to define some minimal set of independent field variables $g^q(x) = g^q(u^p(x))$, the gravitational field, so that it completely defines the stress–energy–momentum tensor of the theory:

$$T^\mu_\alpha(x) = T^\mu_\alpha(g^q(x)).$$

All the other independent variables of the theory are, then, by construction, the "matter fields" $\phi^m(x) = \phi^m(u^p(x))$ of that theory.

V. THE "ACTION EQUALS REACTION" SYMMETRY

After this change of variables, we can now apply the "action equals reaction" symmetry. It is a symmetry which is always present if we have a Lagrange formalism: It is, essentially, the consequence of commutation of functional derivatives: The second order functional derivative does not depend on the order of application of two functional derivatives:

$$\frac{\delta}{\delta u}\frac{\delta}{\delta v} S = \frac{\delta}{\delta v}\frac{\delta}{\delta u} S. \tag{5}$$

Now, the first functional derivative $\frac{\delta S}{\delta u}$ resp. $\frac{\delta S}{\delta v}$ are the Euler-Lagrange equations of motion for u resp. v. Then, the functional derivate for v on the equation of u has the quite obvious meaning of the "action" of v on u. The "reaction" of u on v is, similarly, the functional derivate for u on the equation of v. That means, formula (5) shows that the action of u on v equals the reaction of v on u.

We can now apply this rule to the relation between the preferred coordinates $\mathfrak{x}^\alpha$ and the "matter fields" ϕ^m, which are, by construction, all fields except the "gravitational field" defined by the variables g^q. So, we have

$$\frac{\delta}{\delta \mathfrak{x}^\alpha} S = \partial_\mu T^\mu_\alpha(g^q)$$

By construction, the fields g^q define the gravitational field, and its variables themselves are independent of the matter fields $\phi^m(x)$, so that

$$\frac{\delta}{\delta\phi^m}\frac{\delta}{\delta\mathfrak{x}^\alpha}S = \frac{\delta}{\delta\phi^m}(\partial_\mu T^\mu_\alpha) = 0.$$

Now, the "action equals reaction" symmetry gives

$$\frac{\delta}{\delta\mathfrak{x}^\alpha}\frac{\delta}{\delta\phi^m}S = \frac{\delta}{\delta\phi^m}\frac{\delta}{\delta\mathfrak{x}^\alpha}S = 0$$

But $\frac{\delta}{\delta\phi^m}S$ are the equations for the matter fields. It follows that the Euler-Lagrange equations for the matter fields do not depend on the preferred coordinates $\mathfrak{x}^\alpha$. Which is the Einstein Equivalence Principle.

VI. DISCUSSION

So, the EEP appears to be a simple and straightforward consequence of the Noether theorem, the action equals reaction symmetry, and the identification of the gravitational field with the stress-energy-momentum tensor of the theory following from the translational invariance of the theory in the preferred coordinates and the Noether theorem.

Let's note that, as a side effect of the construction, the "gravitational field" will also have a universal character – usually the states of all fields of the theory will in one way or another contribute to the stress–energy–momentum tensor. This contribution will, in our choice of variables, obtain the form of an interaction of the "matter fields" ϕ^m with the "gravitational field" g^q which defines the stress–energy–momentum tensor.

[1] Schmelzer, I. (2012). A generalization of the Lorentz ether to gravity with general-relativistic limit, Advances in Applied Clifford Algebras 22(1), 203-242, arXiv:gr-qc/0205035

[2] E. Kretschmann, Über den physikalischen Sinn der Relativitätspostulate, A. Einsteins neue und seine ursprüngliche Relativitätstheorie, Ann. Phys. 53, 575-614 (1917)