THE CASE FOR CARDINAL UTILITY

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Abstract. I show that the standard unobservability argument against cardinal utility fails for decision-making under uncertainty, if expectation values of a cardinal utility are maximized. Given an ordinal utility defined by maximization of expected cardinal utility, one can recover the cardinal utility function modulo a linear transformation. Relations between gains and losses appear observable in the same sense as ordinal utility. So the standard Austrian argumentation against cardinal utility is invalid.

The existence of a cardinal utility function does not have to be presupposed: It is sufficient to presuppose a few self-evident rationality principles for an otherwise arbitrary order on the space of probability distributions to prove the existence of an appropriate utility function.

In other words, a decision-making algorithm not equivalent to maximization of expected cardinal utility is irrational.

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1. Introduction

A standard Austrian position is that utility makes sense only as an ordinal, not as a cardinal. This position is defended by Mises:

Berlin, Germany.

> A judgment of value does not measure, it arranges in a scale of degrees, it grades. It is expressive of an order of preference and sequence, but not expressive of measure and weight. Only the ordinal numbers can be applied to it, but not the cardinal numbers. ([2] p.97),

as well as Rothbard:

> Value scales of each individual are purely ordinal, and there is no way whatever of measuring the distance between the rankings; indeed, any concept of such distance is a fallacious one. ([1] p.258).

Other Austrian economists support this position too, in particular Hülsmann:

> . . . ranks are non-extended entities. One can therefore simply not say how high a preference rank is. One can say that a preference rank A is higher than a preference rank B and lower than a preference rank C. That is all. ([13] p.8),

and Hoppe:

> . . . utility can be treated only ordinally; that is, as a rank order on a one-dimensional individual preference scale . . . Apart from their placement on one-dimensional individual preference scales, no quantitative relationship between different goods and different quantities of the same good exists. ([6] p.4),

The rejection of cardinal utility is shared by at least some non-Austrian economists. So Caplan, referring to Kreps [14], writes:

> A utility function is just a short-hand summary about an agent's ordinal preferences, not a claim about "utils." . . . This is why neoclassicals say that the utility function is uniquely defined up to a monotonic transformation. You can rescale any utility function however you like, so long as you re-scale it monotonically.([9] sec. 2.1),

and claims that

> While the exposition of utility theory in undergraduate textbooks may sometimes be open to Rothbards critique of cardinality, neoclassical utility theory is no less ordinal than his own theory [see, e.g., Varian, pp. 9597] ([8] p.827),

referring to Varian who writes

> The only relevant feature of a utility function is its ordinal character ([10] p.95).

On the other hand, Block thinks that

> . . . cardinality lies at the very root of neoclassical economics; indeed, this is one way to distinguish the Austrian School from the mainstream. ([7] p.26)

Whatever, the rejection of cardinal utility is shared by many libertarian economists (Austrian or not), and seems sufficiently interesting and important for them.

The aim of this paper is to show that the rejection of cardinal utility is wrong. Instead, I argue that *a decision-making algorithm which is not equivalent to maximization of expected cardinal utility is irrational.*

Despite this radical claim, I do not have to question the basic philosophical idea of ordinal utility: Every reasonable decision-making can be described by an order on the space of possibly relevant alternative choices, and expected cardinal utility is only a particular method to define such an order. Even more, expected cardinal utility follows the standard scheme of cardinal utility: It defines a cardinal utility $U(\omega)$ on the space of all possibly relevant alternatives ω. Then, the order is simply defined by comparing these utilities. So, it seems, the standard theorem that the cardinal structure does not matter should be applicable: That one can apply an arbitrary nonlinear strong monotonic transformation to the utilities without modifying the order.

But there is a loophole in the proof: If the cardinal utilities $U(\omega)$ are only derived objects, functions $U(\omega, u)$ of other, more fundamental cardinal utilities u, it is not proven that the cardinal structure of the fundamental utilities u does not matter. This loophole is a quite large one: It seems even hard to imagine a sufficiently non-trivial map $U(\omega, u)$, one which reduces the degrees of freedom from "one value for every imaginable situation" to "one value for everything we value", such that the proof could survive, where the cardinal structure would not matter. The proof could survive for a map which fulfills $U(\omega, F(u)) = F(U(\omega, u))$ for every monotonic map $F(.)$, but this is an extremely strong restriction.

So it is at least plausible that the cardinal structure of the real, fundamental values matter, whatever the particular model of fundamental utilities u and the particular map $U(\omega, u)$. And once the proof fails, it is quite plausible that different cardinal values really lead to different decisions. But it seems hard to make this certain – one would have to postulate a particular algorithm $U(\omega, u)$ to prove exactly that the cardinal structure of the u matters, leads to different observable decisions. But so what? People may use different algorithms.

That's not really a good defense. For a critical rationalist it is simply ridiculous. But not everybody is a critical rationalist, and, despite a lot of anti-behaviorist statements which can be found in Austrian writings (look for everything about "verstehen") I share the doubts about the consistency of this anti-behaviourism expressed by Caplan [9], so that I think I have to care to some degree about behaviouristic objections. Fortunately, there is a particular domain where the situation is much easier, where we have an explicit, simple, and nice algorithm, and where there is no rational alternative to this algorithm.

This domain is decision-making under uncertainty, where the $u(x)$ are the utilities of *definite* outcomes, and the algorithm $U(\omega, u)$ simply computes the expectation value $\langle u\rangle(\omega)$.

The algorithm is sufficiently simple to allow a complete clarification of the relevant points. It defines an order $\rhd_u$ between probability distributions by the rule $d\mu \rhd_u d\mu' \;\Leftrightarrow\; \langle u\rangle(d\mu) > \langle u\rangle(d\mu')$. The cardinal structure of the $u(x)$ is important, and one can show not only that the irrelevance proof fails, but even that the $u(x)$ can be recovered from the resulting order $\rhd_u$ modulo a *linear* transformation.

Based on this result, one can also clarify the issue of observability. One can get rid of the remaining freedom of a linear transformation by defining an appropriately normalized "gain function" $r(x)$ by fixing two values $r(x_0) = 0, r(x_1) = 1$. The gain function is, then, observable in the same sense as the resulting order. That means, a theory which postulates that somebody uses a particular function $r(x)$ can be distinguished from the theory that he uses a different one: There are imaginable circumstances where the two theories predict different observable decisions.

Even more, one does not even have to presuppose the use of this algorithm: It is sufficient to presuppose a few self-evident rationality principles for an order $\rhd$ on the space of probability distributions $\mathcal{P}(X)$. This assumption of rationality is already sufficient to construct, for an arbitrary rational ordinal utility $\rhd$, an utility function $u(x)$, so that the order $\rhd_u$ defined by expected cardinal utility algorithm for $u(x)$ is exactly the same as $\rhd$. In other words, somebody who does not use expected cardinal utilities (or an algorithm indistinguishable from using them) violates elementary rationality principles.

What else can be objected against cardinal utility? There are various objections, and I consider some of them (and, as the reader will guess, reject them). Here I want to mention only one major point: Most objections made from proponents of ordinal utility (including all I'm aware of) can be easily rejected by a similarity counterargument: They are applicable to ordinal utility as well.

This is a consequence of the mathematical closeness of the two approaches. Cardinal utility $u(x)$ defines an ordinal utility $\rhd_u$, thus, is exactly a particular example of ordinal utility. What distinguishes cardinal utility, given the results of this paper, are only a few self-evident rationality principles. But the use of some rationality principles can be identified as a part of ordinal utility too: What else is the condition of the transitivity of the order? Clearly not an observable fact – we observe always only one decision, and have no observable information about the order between the rejected choices.

What are the consequences of this for Austrian economics? This problem has to be left to future research, by those more aware of the details of the differences between Austrian economics and their mainstream opponents. What I see is the following: Whenever some position is attacked for its use of cardinal utilities, the argument has to be reformulated in such a way that it does not depend on a rejection of cardinal utility. On the other hand, there may be other cases where the Austrian argument does not survive and cannot be replaced by a better one. In these cases, the Austrian position deserves modification.

This seems to be a loss. But I see also an advantage – the advantage of increased internal consistency of the Austrian approach. In fact, if I read the various explanations of the Austrian "method of understanding", I feel a quite clear contradiction: Understanding by introspection, as well as understanding of other people based on communication with them, clearly tells me that our values and aims have cardinal aspects, and that these cardinal aspects are very important for our decision-making.

Let's finally note that the aim of the paper is mainly pedagogical, addressed to supporters of Austrian economics who believe that there is a strong case against cardinal utility. I do not claim any mathematical novelty here. The very fact that, given some set of rationality assumptions, one can derive some cardinal utility functions such that its expectation value is optimized goes back to von Neumann and Morgenstern [15] chap. 1.

2. The case against cardinal utility

Let's start with the case against cardinal utility. One of the best explanations seems to be the original one given by Mises in Human Action:

> The gradation of the means is like that of the ends a process of preferring a to b. It is preferring and setting aside. It is manifestation of a judgment that a is more intensely desired than is b. It opens a field for application of ordinal numbers, but it is not open to application of cardinal numbers and arithmetical operations based on them. If somebody gives me the choice among three tickets entitling one to attend to operas Aida, Falstaff, and Traviata and I take, if I can only take one of them, Aida, and if I can take one more, Falstaff also, I have made a choice. That means: under given conditions I prefer Aida and Falstaff to Traviata; if I could only choose one of them, I would prefer Aida and renounce Falstaff. If I call the admission to Aida a, that to Falstaff b and that to Traviata c, I can say: I prefer a to b and b to c.
>
> The immediate goal of acting is frequently the acquisition of countable and measurable supplies of tangible things. Then acting man has to choose between countable quantities; he prefers, for example, 15r to 7p; but if he had to choose between 15r and 8p, he might prefer 8p. We can express this state of affairs by declaring that he values 15r less than 8p, but higher than 7p. This is tantamount to the statement that he prefers a to b and b to c. The substitution of 8p for a, of 15r for b and of 7p for c changes neither the meaning of the statement nor the fact that it describes. It certainly does not render reckoning with cardinal numbers possible. It does not open a field for economic calculation and the mental operations based upon such calculation. ([2] p.200)

The example is impressive: What could be the meaning of cardinal utility if I have to decide between Aida, Falstaff, and Traviata? It seems clearly meaningless, or at best metaphorical, to say that I prefer Aida three times more than Falstaff.

But introspection tells me, and I'm sure not only me, that there is more about our preferences than simply the order. Not that we assign numbers named "utilities" to Aida and Traviata. The word "utility" itself is artificial. We care only about differences – gains and losses. And we also do not assign numbers to the gains and losses themself. But we care about their relations.

Say, I prefer Aida to Falstaff and Falstaff to Traviata. Aida, in comparison to Falstaff, is a gain, Traviata in comparison to Falstaff a loss. And everybody agrees that that one cannot assign an absolute number to this gain, as well as to this loss. One can estimate them only in comparison.

But in comparison one can. To estimate them in comparison is something we are doing, even more, we like to do it very much. Alice thinks Aida is excellent, superior, an exceptional masterpiece, and the differences between everything are unimportant. She agrees that Falstaff is a little better than Traviata, but insists that the difference is negligible. Bob heavily disagrees. He accepts that Aida is the superiour masterpiece, but Falstaff is a masterpiece too, and it is their difference which is almost negligible. But to compare Falstaff with Traviata, which deserves to be completely forgotten, is completely inexcusable.

So the values human beings assign to things *have* cardinal aspects. Are these cardinal aspects completely irrational?

3. The defense of cardinal utility

I claim that they are not. Instead, they are completely rational devices for decision-making in complex situations. You have to decide between 15r and 7p. Then you have to decide between 8p and 23q. And then between 4r and 6q. You are sure your choices will be consistent? You may not care, like Grimm's "Hans im Glück", but most people prefer to care. Having cardinal values for p, q, and r is extremely helpful, and having them surely matters – that's the very reason to have them – even if one applies them only as a first, rough approximation, and leaves oneself the freedom to decide differently, following spontaneous whims.

But, while it is rational to have such values, it is also often rational not to follow the simple algorithm to maximize these values. Arguing that most people follow some algorithm approximately, or that rational people should follow some algorithm approximately, necessarily leaves a lot of room for disagreement. Fortunately there is a domain of decision-making where the situation is much easier, where the rules of rational behaviour are extremely simple, simple enough to formulate them as exact mathematical rules.

This domain is rational decision-making under uncertainty.

So let's consider a variant of the choice between the three operas where uncertainty is involved. I have to make a choice between two alternatives: One decision d_0 gives a result with certainty – I can take a ticket for Falstaff (x_0). The alternative decision $d_\pm$ is to take a ticket from my friend, who has it at home, but is not sure if it is for Aida (x_+) or Traviata (x_-). All I have is some probability p_+ that it is for Aida and $p_- = 1 - p_+$ that it is for Traviata. (See sect. 12 about the question if it makes sense to define such a numerical value.)

Let's assume now that Mises is right and there is not more to say about my preferences than the order: I prefer Aida, and my second choice would be Falstaff. If I denote my preferences with an ordering relation $\succ$, then I can write my preferences as $x_+ \succ x_0 \succ x_-$. So, what we assume here is the simple concept of ordinal utility that what is ordered are definite states – Aida, Falstaff, and Traviata – without any cardinal aspects of these preferences.

So what will be my decision, given these preferences together with the probabilities $p_+ + p_- = 1$?

The answer is that the information about my preferences, as given by the order $\succ$, is not sufficient to make a rational choice.

Let's see why. First, let's recognize that the decision depends in a nontrivial way on the probabilities. Indeed, let's simply look at extremal values of the probabilities. First, assume that $p_+ = 0$, that means, that it is impossible that the ticket is for Aida. It is certainly for x_-, Traviata. Clearly, I prefer the ticket for Falstaff and choose d_0. Let's look now at the other extreme $p_+ = 1$ that the ticket is certainly for x_+, Aida. Once I prefer Aida to Falstaff, and the risk to end up with Traviata does not exist, I make the choice $d_\pm$ and take the ticket from my friend.

So, in the two extremal cases $p_+ = 0, 1$, I will make different choices. It is also clearly rational that an increasing probability p_+ for Aida makes the risky decision $d_\pm$ more attractive. So, this purely qualitative consideration gives a clear result: There will be some critical probability r where I change my choice from d_0 for $p_+ < r$ to $d_\pm$ for $p_+ > r$. A probability where I consider the chance to get the Aida ticket as worth to accept the risk to end up with the Traviata ticket. Everything else would be irrational, inconsistent, violate common sense.

But what about the value r? How is this value defined? The answer is that this depends. For different people, with different opinions about the three operas, there will be different values of r, even if they all have the same ordering preferences as $x_+ \succ x_0 \succ x_-$.

Indeed, this is the point where Alice and Bob disagree. Let's consider the probabilities as fixed by $p_+ = p_- = \frac{1}{2}$ and compare what Alice and Bob think about the choice. Above share the same order of preference $x_+ \succ x_0 \succ x_-$. But Alice clearly prefers Aida, but she is almost indifferent between Falstaff and Traviata. So Alice will clearly prefer the risky choice $d_\pm$, because it gives her at least a chance to get a ticket for Aida. And this will remain her choice even for p_+ much larger than $\frac{1}{2}$. That means, her value of r will be quite large, close to 1.

Instead, for Bob the difference between Aida and Falstaff is not that important, but Traviata is a no go for him, he has already seen it and not liked it at all. So, Bob will clearly prefer the sure choice d_0 of Falstaff. And, given his interest, he will prefer d_0 not only for $p_+ = \frac{1}{2}$, but even for much lower p_+. In other words, his value of r will be quite small, close to 0.

Thus, the interests of Alice and Bob are clearly different, and this results in different choices. Given their values, every other choice would be completely off for them. Clearly they act differently because they have different values.

If cardinal utilities would not matter, their interests, as far as they are relevant for their decision-making process, would be identical. Their cardinal utilities lead to the same ordering of interests: $x_+ \succ x_0 \succ x_-$. So the thesis that cardinal utilities do not matter is unable to explain their different choices.

Instead, using cardinal utilities $u(x)$ for the three operas leads to a simple and natural algorithm for decision-making: One has to compare $u(x_0)$ – the utility of the Falstaff ticket – with $p_+u(x_+)+p_-u(x_-)$ – the expectation value of the utility of the ticket of my friend. One can easily see that this simple algorithm fulfills all the conditions of rational choice we have identified in our consideration: The preferences in the extremal cases, the increasing reasonableness to risk with increasing chances to get the Aida ticket, the existence of a single critical probability r where the decision changes.

One could object that Alice and Bob do not have such "utilities" in mind. Indeed. What they have in mind is what matters. Utilities make sense only modulo a linear transformation $F(u) = au + b$; $a > 0$. One can set two of the three numbers to arbitrary values, say, $u(x_+) = 1$, $u(x_-) = -1$. So there is one real number which remains after this normalization, and this number, $u(x_0)$, defines the size of the loss of getting Traviata instead of Falstaff, in comparison with the gain of getting Aida instead of Falstaff. It is this cardinal number, this relation, which is important for decision-making, which defines the critical probability r, and about which people care.

4. Probabilistic ordinal utility

"You have completely misunderstood the concept of ordinal utility," objects the proponent of ordinal utility, "you argue about preferences between x_0 and the uncertain state $d\mu$ described by a probability distribution. Of course, the order between x_0 and $d\mu$ is not defined by the order between the definite states x_-, x_0, and x_+. $d\mu$ is simply a different state. The concept of ordinal utility assumes that there is an order between all the states which are relevant for decision-making, and in the example it is the state $d\mu$ which is relevant for decision-making, thus, the order between x_0 and $d\mu$ has to be defined. So, once we consider decision-making under uncertainty, the order $\succ$ on the definite states X is not sufficient, and one has to consider an order on the space of probability distributions $\mathcal{P}(X)$."

Fair enough. But I object that the point of my argument has been misunderstood. The point about the order $x_+ \succ x_0 \succ x_-$ as being insufficient was to clarify that for the decision-making algorithm which I have described – the algorithm based on cardinal utilities $u(x)$ for definite states, which maximizes the expected utility – the cardinal structure matters. If it would not matter, then the corresponding order $\succ$ on the set of definite states X, defined by $x \succ x'$ if $u(x) > u(x')$, would be sufficient.

So, if we presuppose that this particular algorithm is used under conditions of uncertainty, the cardinal structure of the $u(x)$, which are defined on the definite states $x \in X$, matters.

It is not at all my intention to argue that ordinal utility is impossible.

But the concept of ordinal utility which is viable is a different object – it is an order not on the space of definite states X, but on the much larger space of probability distributions $\mathcal{P}(X)$ defined on this space. These spaces should not be mingled, and to avoid confusion I will use different denotations for them. I will use x with various indices to denote definite states, elements of X, but $d\mu$ or $d\omega$ with various indices for states defined by probability distributions, elements of $\mathcal{P}(X)$. But definite states are also particular examples of distributions – delta-distributions. To denote the distribution localized in x I use the denotation $\hat{x}$, so that $x \in X$ and the corresponding $\hat{x} \in \mathcal{P}(X)$ will be distinguished. Then I use lower letters like $u(x)$, $r(x)$ to denote real functions on definite states X, for such functions on $\mathcal{P}(X)$ I will use uppercase letters like $U(d\mu)$. Last but not least, I use the "thin" symbol $\succ$ for an order on the space X but the "fat" symbol $\rhd$ for an order on the much larger space $\mathcal{P}(X)$.

So I accept that decision-making under uncertainty can be described by a *probabilistic* ordinal utility – by an order $\rhd$ on $\mathcal{P}(X)$.

Even more, I accept that this concept is universally applicable, that it is the only reasonable, rational one – every reasonable decision-making process under uncertainty has to be equivalent to the definition of such an order $\rhd$ on $\mathcal{P}(X)$.

5. The decision-theory algorithm

And this does not contradict at all my claim that a decision-making algorithm which is not equivalent to maximization of expected cardinal utility is irrational: This algorithm is simply a particular case, a particular method to define such an order $\rhd_u$ on $\mathcal{P}(X)$ given a cardinal utility $u(x)$ on definite states $x \in X$:

Theorem 1 (decision-theory algorithm)**.** *Assume we have given an arbitrary deterministic cardinal utility – a bounded measurable*[1] *function $u(x) : X \to \mathbb{R}$. Then the binary relation $\rhd_u$ on $\mathcal{P}(X)$ defined by*

$$d\mu \rhd_u d\mu' \quad \Leftrightarrow \quad \langle u\rangle(d\mu) > \langle u\rangle(d\mu') \qquad \text{with} \qquad \langle u\rangle(d\mu) = \int_X u(x) d\mu(x) \quad (1)$$

[1]Non-measurable functions are artefacts of mathematical set theory which play no role at all in applications. We have included this word only for accuracy.

defines a probabilistic ordinal utility – a linear order $\rhd_u$ *on* $\mathcal{P}(X)$.

One may object against such algorithms that real people do not follow such algorithms. But I do not even claim that real people actually use this algorithm – they clearly don't compute integrals. The point of considering this algorithm is that it is a rational one. And, except for the many occurrences where people err, they behave at least approximately rational, thus, to consider rational algorithms is also useful as an approximate theory of human behaviour. But for the purpose of this paper, such objections are irrelevant also for another reason – they apply equally to ordinal utility too. Real people also behave in ways which violate the transitivity condition of an order. Moreover, they often engage in wishful thinking, that means, their evaluation of the outcomes of the decisions is influences by their aims and wishes – an effect which is not described by the model of ordinal utility, where the order describes the aims, and the outcomes of the decisions are computed independently.

The reason to consider this algorithm is, therefore, not that people actually use it (even if it is important that it is in agreement with common sense and approximately followed) but that it is a rational algorithm.

6. The mathematical argument against cardinal utility

The first and most important point is that for this algorithm the unobservability argument against cardinal utility fails. So let's see where this seemingly simple mathematical proof fails.

Let's start with the situation where the proof is correct – the case of decision-making under the condition of certainty.

Under certainty, the decision-theoretic algorithm simplifies. We have to compare only definite outcomes, so we don't have to consider expectation values: $\langle u\rangle(\hat{x}) = u(x)$. So the order $x \succ x'$ is simply defined by $u(x) > u(x')$.

Now let's modify the cardinal structure preserving only the order. The way how to do this is to apply an arbitrary monotonic transformation. Such a transformation is defined by a strong monotonic function $F(u): \mathbb{R} \to \mathbb{R}$. All what is required is the condition of strong monotonicity: $u > u' \Rightarrow F(u) > F(u')$. Such a transformation can completely change those aspects we have characterized as important – the relations of gains and losses. The utility of Falstaff may be shifted down close to Traviata, as for Alice, or shifted up close to Aida, as for Bob. So the relation between the gain of getting Aida instead of Falstaff and the loss of getting Traviata instead can be shifted from very large for Alice to very small for Bob.

The result is that the deterministic utility, as defined by the decision-theoretic algorithm, remains the same: $\succ_u$ is the same order as $\succ_{F(u)}$. And, indeed, Alice and Bob agree about this order: Aida is better than Falstaff, and Falstaff better than Traviata. Up to now, there is no contradiction between this proof and our example. And also none with Mises's original argument: If confronted with a simple choice between the three tickets, they would make the same decisions. For decision-making under certainty their different opinions do not matter.

I claim that for decision-making under uncertainty this is no longer true.

But the proof can be easily extended to arbitrary probabilistic utilities. So let's consider a general probabilistic utility $U(d\mu): \mathcal{P}(X) \to \mathbb{R}$, which defines a probabilistic order $\rhd_U$ by the rule that $d\mu \rhd_U d\mu'$ if $U(d\mu) > U(d\mu')$. We can apply the same general nonlinear monotonic function $F(.)$ to $U(d\mu)$ and obtain the same result: The order does not change, $\rhd_U$ is exactly the same order as $\rhd_{F(U)}$.

And the particular probabilistic utility $\langle u\rangle$ defined by the expectation value is not more than a particular choice of a probabilistic utility $U(d\mu) = \langle u\rangle(d\mu)$. The theorem has been proven for all $U(d\mu)$, so it holds also for $\langle u\rangle(d\mu)$. Case closed.

Wrong. Case not closed. The claim is that the cardinal structure of $u(x)$ matters. So one would have to prove that $u(x)$ gives the same order as $F(u(x))$. That means one would have to prove that $\langle u\rangle$ gives the same order as $\langle F(u)\rangle$. This is quite different from the claim that $\langle u\rangle$ gives the same order as $F(\langle u\rangle)$, what has been proven. The two utilities $\langle F(u)\rangle$ and $F(\langle u\rangle)$ are in general different: $F(\langle u\rangle) \neq \langle F(u)\rangle$.

The proof goes through only in a particular case – that of a linear strong monotonic function $F(u) = au + b;\ a > 0$. For linear F, we have $F(\langle u\rangle) = \langle F(u)\rangle$, because the expectation value is a linear functional. And so we can complete the proof, and find that in this case $\rhd_u$ is really the same order as $\rhd_{F(u)}$. But for non-linear F the proof fails.

In fact, our example illustrates the difference. The utilities of Alice and Bob can be transformed into each other by some non-linear strong monotonic F, which shifts the utility $u(x_0)$ for the Falstaff ticket up and down between -1 and 1 but leaves the other two values unchanged. Alice and Bob above follow the decision-theoretic algorithm, maximize their expected utility. As the result, they make different choices for the case of $p_+ = \frac{1}{2}$, thus, their order $\rhd$ on $\mathcal{P}(X)$ is different. So the monotonic transformation F has changed the order, and $\rhd_{Alice}$ and $\rhd_{Bob}$ appear different.

This is already a very important result, it shows that the standard objection of Austrian economics *against* cardinal utility *fails*: *If* one presupposes the use of the decision-theory algorithm, and applies it in situations of *uncertainty*, then the standard argument that the cardinal structure does not matter is invalid.

7. The cardinal structure

We have seen already that some part of the argument against the cardinal structure remains nonetheless valid: The real numbers $u(x)$ themself do not matter. One can apply an arbitrary order-preserving *linear* transformation $F(u) = au+b;\ a > 0$ without modifying the resulting order, and, therefore, without modifying any observable decision. The structure which is preserved by such linear transformations is named by mathematicians an *oriented affine structure.*

So, cardinal utility may be not an optimal choice for naming this concept, once the numbers $u(x)$ themself do not matter. But this does not lead to any danger of confusion, because nobody claims that the numbers $u(x)$ themself do matter. And therefore I do not propose to rename cardinal utility into something else, like affine utility. It is simply necessary to clarify that "cardinal utility" does not mean that the numbers $u(x)$ themself are important, but only those things are important which remain unchanged if one applies an order-preserving linear transformation $F(u) = au + b;\ a > 0$. After this clarification, one can use the established notion of cardinal utility.

But, wait: Up to now we have only shown that the standard argument against cardinal utility fails – we cannot prove that $F(u)$ and u are indistinguishable. But it does not yet follow that the cardinal structure really matters. Maybe there is another proof? Maybe only some part of the structure is important? The following theorem (proven in app. A) excludes this possibility:

Theorem 2 (reconstruction)**.** *Assume two deterministic cardinal utilities u, u' define by (1) the same probabilistic ordinal utility $\rhd$.*

Then there exists such $a > 0, b \in \mathbb{R}$ so that $u'(x) = au(x) + b$.

In other words, whatever remains invariant for arbitrary order-preserving linear transformations $au + b$; $a > 0$ is already *completely* defined by the probabilistic ordinal utility $\rhd_u$.

What are these things which remain unchanged if one applies such a linear transformation? First, let's consider not the utilities themself, but utility differences. Utility is anyway an artificial, scientific word, not used in everyday language, but for utility differences we already have words in common language – gains and losses. These are already more invariant – translations $F(u) = u + b$ do not change them.

Common sense also knows that gains and losses are not some absolute numbers, that only relations between them are important. But relations between utility differences are already invariants of linear transformations:

$$\frac{u(x_1) - u(x_2)}{u(x_3) - u(x_4)} = \frac{u'(x_1) - u'(x_2)}{u'(x_3) - u'(x_4)} \qquad \text{if} \qquad u'(x) = au(x) + b. \tag{2}$$

One does not have to consider all these relations. One such relation $r(x)$ for each state $x \in X$ is sufficient. For this purpose let's fix one arbitrary reference state $\hat{x}_0$ (say my actual state here and now), and another, different one $\hat{x}_1$ (say, the same actual state, but with me having 1€ more). Their difference $u(x_1) - u(x_0)$ plays the role of a unit of gain. Then the utility $u(x)$ of every other state is already uniquely defined by the invariant relation $r(x)$ and the two arbitrary utilities $u(x_0)$ and $u(x_1)$:

$$r(x) = \frac{u(x) - u(x_0)}{u(x_1) - u(x_0)}; \qquad u(x) = r(x)(u(x_1) - u(x_0)) + u(x_0). \tag{3}$$

This function $r(x)$ can be used as an utility function itself, normalized by the conditions $r(x_0) = 0$, $r(x_1) = 1$. In other words, the utility function can be fixed uniquely by a simple normalization condition. We will name this function *"gain function"*.

8. The question of observability

But does it really follow from our reconstruction theorem 2 that the gain function is *observable*?

Given that, in principle, aims can change in time, all the sufficiently certain observables about our aims are restricted to a single decision. This single decision is preferred to all alternatives. This is certainly not much. And it is clearly not sufficient to identify, by observation, the whole structure of our decision-making database. The simplified assumption of stability of the aims over time gives some more observable data, nonetheless even with this assumption one cannot recover the whole structure.

But there is a weaker notion of observability – observability in principle. The human being who uses our algorithm with some function $r(x)$ acts in a predictable way in every imaginable situation. And now we can ask which part of the information can be extracted in principle by observation. Here, two sets of aims are indistinguishable in principle if all decisions, in all imaginable situations, are identical. Whenever where is a difference for at least one imaginable situation, the two sets of data are distinguishable by observation in principle.

In Popper's language, a structure for decision-making is observable in principle if the claim that two different structures are identical is falsifiable by observation.

If we restrict ourself to this concept of observability, the gain function $r(x)$ is observable. Indeed, assume we have two different gain functions $r(x)$, $r'(x)$. It follows from the reconstruction theorem 2 that the resulting probabilistic orders $\rhd_r$, $\rhd_{r'}$ have to be different. Once they have to be different, there will be at least one pair $d\mu$, $d\mu'$ so that $d\mu \rhd_r d\mu'$ and $d\mu \lhd_{r'} d\mu'$. So, in a situation where one has to choose between $d\mu$ and $d\mu'$ one will decide differently.[2]

Empiricists or behaviorists may not like such a restricted notion of observability, but this is certainly nothing I would care about.

But even if one follows empiricism or behaviourism, one cannot extract from the issue of observability an argument against cardinal utility in favour of ordinal utility. In fact, the complete order $\rhd$, which is presupposed by ordinal utility theory, is observable only in the same weak sense as cardinal utility. Or, in other words, the reconstruction theorem 2 transforms the question of observability of the gain function $r(x)$ into the analogical question of observability of the probabilistic ordinal utility $\rhd_r$.

So, the cardinal structure, as defined by the gain function $r(x)$, is observable in the same sense as the order $\rhd$ itself is observable. Those who accept ordinal utility as a valid concept cannot object against cardinal utility using an unobservability argument.

9. Rationality principles for probabilistic ordinal utilities

Up to now I have presupposed the existence of an utility function $u(x)$ so that the order $\rhd$ is equivalent to $\rhd_u$. As a critical rationalist, I have no problem making such theoretical presuppositions. There is sufficient support from common sense as well as introspection that human decision-making under uncertainty is based on the consideration of relative gains and losses. Nonetheless, the question is interesting if there are other reasonable ways to define probabilistic ordinal utilities.

There are, clearly, such orders $\rhd$ which cannot be obtained as $\rhd_u$ for some utility $u(x)$. Imagine a number fetishist, who hates situations where his chances to win are too close to the "evil probability" of 0.666, and prefers, whatever the outcomes, a situation where his chances to win are different – smaller or better does not matter. This defines, of course, also some order $\rhd$ on $\mathcal{P}(X)$. And there is obviously no utility function $u(x)$ which would recover such an order as $\rhd_u$.

But this order is quite obviously irrational. To exclude such clearly irrational probabilistic ordinal utilities, let's define now some rationality principles.

For proponents of ordinal utility there is not much to object against the very idea to presuppose some rationality principles: Ordinal utility also presupposes some rationality principles, in particular the principle of transitivity – if one prefers

[2]This "proof" is a little sloppy – the orders would be different already if $d\mu \rhd_r d\mu'$ and $d\mu \triangleq_{r'} d\mu'$, and in this case above could choose $d\mu$. So one has, in fact, to identify some $d\mu, d\mu'$ with $d\mu \rhd_r d\mu'$ and $d\mu \lhd_{r'} d\mu'$ explicitly. But this is not difficult having in mind the constructions given in app. A. Taking an x where $r(x) \neq r'(x)$, a value $\tilde{r} = \frac{1}{2}(r(x) + r'(x))$ between them, and comparing $\hat{x}$ with $d\omega_{\tilde{r}}$ would to the job.

A to B and B to C, one has to prefer A to C too. Human psychology knows that this condition is often violated by real human beings, so this is a nontrivial restriction too.

But let's consider now the rationality principles I propose for probabilistic ordinal utilities:

Principle 1 (preference for better chances)**.** *Assume* $\hat{x} \rhd \hat{x}'$.
Then $(p\hat{x} + (1-p)\hat{x}') \rhd (q\hat{x} + (1-q)\hat{x}')$ *if* $p > q$.

In simple words, in a state of uncertainty between two states I prefer to have higher probability for the outcome I prefer.

Principle 2 (preference for better outcome)**.** $\forall d\mu, d\mu', d\omega \in \mathcal{P}(X)$: *if* $d\mu \trianglerighteq d\mu'$ *then* $pd\mu + (1-p)d\omega \trianglerighteq p'd\mu + (1-p)d\omega$ *(resp. for* $\trianglelefteq$*).*

That means, in a state of uncertainty I prefer a situation where, everything else fixed, one possible outcome is replaced by another, preferable one.

Above principles are so self-evident and obvious that it does not seem to make sense to justify them by anything else. This is the classical case where further justification only increases doubt – the axioms one starts with simply cannot be more obvious and more self-evident.

Except, maybe, the observation that already the very phrase "better chance" implicitly presupposes this principle: The meaning of "better chance" is a higher probability to win, thus, presupposes a preference between the possible outcomes. But that such a higher probability to win it is named "better chance" assumes that it is preferable.

It is interesting to compare these principles with similar approximate principles in a deterministic world. Say, a deterministic analog of the preference for better chances could be a principle of preference for more items of something we like. Say, if we prefer to have 1000€ in comparison to nothing, we should prefer also to have x€ in comparison to y€ is $x > y$. This principle is a nice approximation, and we follow such principles quite often, but it is certainly not a strong law of rational reasoning, and there are a lot of exceptions: Think about the higher necessities for security, higher taxation by states and so on. Often enough state regulations make exceptions for incomes below x €, which makes it preferable to have $(x-1)$€ instead of x €. So what is the difference? Why is the "preference for more items" only an approximation which sometimes fails, while I propose to accept the "preference for better chances" as an absolute, exception-free law of rational reasoning?

The difference is that the "probabilistic states" are not really states. Different from states of the world, "states" of probability theory appear only in human thoughts and theories. Even if we follow an objective, frequentist interpretation of probability, we never appear in a frequency, but always only in a definite state. Now, the real states involved in the principle of preference for better chances are the same in all the "probabilistic states" compared there – they are the two states x_0, x_1. There is also no preference for risk involved. We are always, conceptually, in the same state of uncertainty, with the same possible outcomes x_0 and x_1, only with different degrees of plausibility for each of the two outcomes. So the things which distinguish the alternatives are parts of our thoughts, completely under our control, and no unexpected consequences of the really different states with different amounts of € can play any role.

Therefore I think these two principles clearly deserve the status of *principles of rationality*.

There is also another way of understanding the role of these two principles: They connect two structures – on the one hand, the order $\triangleright$, and, on the other hand, the affine structure of the space of probability distributions $\mathcal{P}(X)$. Indeed, above principles make assumptions about the order between some states $d\mu \trianglerighteq d\mu'$ and make conclusions about the order between some affine combinations of such states. So these principles are compatibility principles: An order $\triangleright$ on $\mathcal{P}(X)$ is compatible with the affine structure of $\mathcal{P}(X)$ if it fulfills the rationality principles.

Recognizing this nature of the rationality principles as compatibility principles makes clear that we have to presuppose also another set of rationality principles – the principles of probability theory. They are used in our proofs in an implicit way if we handle sums like $\sum p_i \hat{x}_i$ as if the rules of usual arithmetics are applicable, so that we, in particular, don't have to care about about brackets.

For a justification of the complete set of mathematical rules of probability theory as principles of rationality see, for example, Jaynes [11].

10. The continuity principle

Let's finish with a principle which is about the compatibility between the order and topology:

Principle 3 (continuity)**.** *If* $\forall i\ d\mu_i \trianglerighteq d\mu$ *then* $\lim_{i\to\infty} d\mu_i \trianglerighteq d\mu$ *if this limit exists (resp. for* $\trianglelefteq$*).*

This principle which makes sense in the deterministic situation too. It is simply a principle of connection between topology and order, unspecific to the situation of uncertainty we consider here. Whenever we have a space with some topology together with some order, it is straightforward to require this principle. So initially I was quite uncomfortable with the classification of this principle: I had to mention it, because I had to use it in the proof (see app. A). But to list it among the special rationality principles related with the specific situation of decision-making under uncertainty?

But then, thinking about an objection made by an anonymous referee to a first version of this paper (thanks), I recognized that this objection referred to a quite common irrationality, an irrationality which is specific to decision-making under uncertainty, and an irrationality which violates exactly the continuity principle.

It is an extremal version of the quite rational aversion against uncertainty. I moderately prefer $\hat{x}_1$ to $\hat{x}$, but I'm extremely afraid of $\hat{x}_0$. Now I have to decide between $\hat{x}$ and a situation with uncertainty between $\hat{x}_1$ and $\hat{x}_0$. The rational decision is to have some extremely small probability p_0 for x_0 which I would ignore as irrelevant. The irrational modification is that there is no, however small, probability p_0 of x_0 I'm ready to tolerate, except absolute certainty that it does not happen, $p_0 = 0$. But that means I violate the continuity principle: Let's use a sequence $d\mu_i$ with $\hat{x}_1$ as the limit but non-zero probabilities $p_0 > 0$ for $\hat{x}_0$. Then we have $d\mu_i \triangleleft \hat{x}$ but $\lim_{i\to\infty} d\mu_i = \hat{x}_1 \triangleright \hat{x}$, violating the continuity principle.

It seems that this error is so natural and common that it is worth to justify the claim that it is irrational. But, once you prefer $\hat{x}_1$, you may agree to pay me something to switch from $\hat{x}$ to $\hat{x}_1$. Then, our world is uncertain, so a more careful evaluation of the outcome of this exchange will always reveal that there is some non-zero probability of x_0 involved. I try a little and explain you this completely irrelevant possibility. Recognizing this, you ask me to switch back. I agree – if you want only half the money back.

This last decision is irrational, because there are also other, equally irrelevant, possibilities to end in x_0 in your final state x. You don't care about them, because you simply cannot – there are, in every situation, far too much possibilities to care about them all. It is this necessity to ignore a lot of small probabilities which makes it irrational to care about some particular very small probability, one which is much smaller than the ones you ignore necessarily in your everyday behaviour.

This irrational human fear of irrelevant, extremely small probabilities is in fact often misused. The justification of increasing police and state power based on single, exceptional criminal cases comes to mind.

So I think the continuity principle also deserves the classification as an *principle of rationality*.

Now, our list of principles of rationality is already finished. And it seems reasonable to presuppose them, to reject all those ordinal utilities which violate them as irrational. So we will name probabilistic ordinal utilities $\rhd$ which fulfill these rationality principles *rational* ordinal utilities.

11. How to identify rational ordinal utilities?

A rational probabilistic ordinal utility is obviously a quite complex object: It is defined on a quite large, functional space $\mathcal{P}(X)$, and it has to fulfill not only the usual axioms of ordering, like transitivity, but also these additional rationality principles. It would be nice to have methods to guarantee that our ordinal utilities are rational, ways to distinguish rational from irrational ordinal utilities.

One such method is provided by our decision-making algorithm of standard decision theory. The resulting probabilistic ordinal utility $\rhd_u$ appears to be rational:

Theorem 3 (rationality)**.** *Every order $\rhd_u$ created by the decision-theory algorithm defined in theorem 1 is rational.*

The proof of this theorem is straightforward: The principle of continuity follows from the continuity of the expectation value $\langle u \rangle$, and the principles of rationality from the linearity of $\langle u \rangle$.

So we have already a remarkably simple way to define rational probabilistic ordinal utilities – using arbitrary deterministic cardinal utilities $u(x)$.

So what? Maybe there are other interesting rational probabilistic ordinal utilities, which cannot be obtained in this way? No, there are no such other ordinal utilities. We have the following theorem about this:

Theorem 4 (existence)**.** *Assume a probabilistic ordinal utility $\rhd$ on $\mathcal{P}(X)$ for a finite set X fulfills the principles of preference for better outcome, preference for better chances, and continuity. Then there exists a deterministic cardinal utility function $u : X \to \mathbb{R}$ which gives $\rhd$ as $\rhd_u$ by* (1).

The proof will be given in appendix A. I have restricted myself here to the technically simpler case of finite sets. Once human beings are able to distinguish only a finite number of different states of the world, this seems completely sufficient for economics.

In fact the rationality principles we have used are sufficient to construct the gain function $r(x)$:

Theorem 5. *Let* $\rhd$ *be an arbitrary rational probabilistic ordinal utility on* $\mathcal{P}(X)$ *and* $x, x_0, x_1 \in X$ *be three arbitrary states with* $\hat{x}_1 \rhd \hat{x}_0$*. Then* $\rhd$ *uniquely defines a rational function* $r(x)$ *– the relative gain – such that* $r(x_0) = 0$*,* $r(x_1) = 1$*, and every deterministic ordinal utility function* $u(x)$ *which gives* $\rhd$ *as* $\rhd_u$ *fulfills the condition*

$$u(x) = r(x)(u(x_1) - u(x_0)) + u(x_0). \tag{4}$$

This gain function defines an order $\rhd_r$, and the methods of the proof of theorem 4 are sufficient to prove that this order is identical to the original one for all the definite states $\hat{x}$ as well as their finite linear combinations $d\mu = \sum_i p_i \hat{x}_i$. Given that we have, among our rationality principles, also the continuity principle, it seems at least not hopeless to extend this to arbitrary probability distributions. But this would be a question of purely mathematical interest and beyond the scope of this paper.

Thus, the results are quite clear: Every rational probabilistic ordinal utility $\rhd$ is equivalent to one defined by a cardinal utility $u(x)$.

12. What if frequentists are right?

In section 3 I have described the reason for uncertainty in terms of a friend who "is not sure". I have, then, characterized my uncertainty by a numerical value p_+. Does this make sense? Many Austrians would object and agree with Hoppe, who writes:

> Frank H. Knight and Ludwig von Mises are entirely correct in insisting that the use of numerical probabilities is impossible in our daily endeavors of predicting our own and our fellow mens actions. As Richard von Mises, the originator of the frequency interpretation of probability, has unambiguously stated: the application of the term probability to a single event is "utter nonsense." ([5] p.19)

What to do? Open Pandora's box of the discussion between frequentist and subjectivist interpretations of probability? This is certainly an interesting question,but beyond the scope of this paper. Fortunately, the decision between frequentism and subjectivism does not influence the main result of this paper. The argument in favour of cardinal utility survives.

Indeed, let's assume for the sake of the argument that the frequentists are right. That means, in the scenario as I have presented it in section 3, to assign a numerical probability p_+ is utter nonsense. Nonetheless, the argument in favour of cardinal utility can be easily saved. I simply have to consider a slightly different scenario.

In the modified scenario, my friend has two tickets – one for Aida, one for Traviata. It would be fine if he would prefer Traviata – he would take the Traviata ticket and give me the Aida ticket. Unfortunately, he prefers Aida too. On the other hand, he wants to get rid of one of the two tickets anyway, and, moreover, he likes to gamble, and so he makes me the offer to use coin flipping to decide who gets the Aida ticket. So we are now in a situation where frequentists no longer object to the appropriateness of the use of numerical probabilities. Thus, the argument goes through, independent of the question of frequentist vs. subjectivist probability.

But what if some frequentists are clever enough to recognize that I have cheated them? I have to confess that, indeed, I know the arguments of chap. 10 of [11] against the objectivity of coin flipping. No problem. They are frequentists, so, by definition, they have to believe in some objective probability – if not of coin flipping, then of something else. Say, radioactive decay. Whatever it is, it will be easy to modify the scenario in such a way that a device based on this objective probability will be used to decide who gets the Aida ticket. Of course, the scenario becomes even less plausible if the proposal is to use the number of clicks of a Geiger counter to decide about the Aida ticket. But this does not diminish its conceptual value – thought experiments are not considered because of their plausibility.

So, the case for cardinal utility stands even if we accept the frequentist's rejection of numerical probability for everything outside natural sciences. The argument, therefore, does not depend on the problem of interpretation of probability.

All one needs, and what is (of course) presupposed in the proofs, is the *mathematical apparatus* of probability theory, in particular the affine structure of the space $\mathcal{P}(X)$ of probability distributions.

13. The similarity counterargument

If one considers other possible objections against the concept of cardinal utility, as presented here, one observes that for almost every argument the similarity between ordinal and cardinal utility is close enough to allow for a counterargument of similarity: The argument against cardinal utility cannot be used by proponents of ordinal utility because it is applicable in the same or a slightly modified way against ordinal utility too.

I have applied this similarity counterargument already two times: First, for the question of observability, where my argument was, essentially, the reduction of the problem of observability of the gain function $r(x)$ to the question of observability of the probabilistic ordinal utility $\rhd_r$. Second, for the question of general, philosophical acceptability of the presupposition of principles of rationality, where I have argued that the concept of ordinal utility also presupposes, with the principle of transitivity of an order, a rationality principle.

I could have applied it as well in the case of the frequentist's objection, if I would have seen a necessity.

There are also other situations where this similarity counterargument is applicable.

So the decision-theoretic algorithm presupposes also some other rationality principles which have not been mentioned up to now. In particular, a principle of avoidance of wishful thinking: There is a part of the algorithm which computes the probability distributions $d\mu(d)$ for the outcomes of our decisions, and there is a completely separate part which describes our aims and wished – the utility $u(x)$. Wishful thinking would mingle these parts, allow an influence of our aims and wishes $u(x)$ on the computation of the probability $d\mu(d)$ of the outcomes of our decisions. So, the principle of avoidance of wishful thinking is also presupposed by the decision-theoretic algorithm. But it is, of course, also part of ordinal utility. Ordinal utility also separates the computation of the outcome $d\mu(d)$ of our decisions and the order $\rhd$ which describes our aims. So any objection against the reasonableness of this principle made by a proponent of ordinal utility could be rejected by the similarity counterargument too.

The very closeness of the mathematical apparatus of above concepts suggests that this counterargument will be applicable in other situations too, for other imaginable arguments I'm not aware of. Indeed, the algorithm of decision theory is, because of the map $u(x) \to \rhd_u$, exactly a particular example of the general algorithm of ordinal utility. And, because of the existence theorem 4, the concept of rational probabilistic ordinal utility is mathematically even identical to the decision-theoretic algorithm. So, it seems, the only possible way to avoid the similarity counterargument would be to argue against the particular principles of rationality we have proposed.

Note, again, not against the very concept of presupposing rationality principles, because ordinal utility theory presupposes such principles too – I have mentioned transitivity of the order and the avoidance of wishful thinking. The objection should be specific, directed against the particular rationality principles I have proposed here.

I see no reasonable chance for such objections.

14. So what is preferable?

Technically, the main result of this paper is an equivalence result: The model of rational probabilistic ordinal utility – an order $\rhd$ on $\mathcal{P}(X)$ which fulfills some rationality principles – is mathematically equivalent to the decision-theoretic model of ordinary cardinal utility – the order defined by the expectation value of a function $u(x)$ on the space of definite states X.

So there remains the question which of the two approaches is preferable.

Here, there are several points to be made in favour of cardinal utility.

First, the agreement with what common sense and introspection tells us about how human decision-making really works. Probability distributions are nothing real people think about in real decision-making. In fact only a few scientists know what "probability distribution" means, and even those who know will hardly think in terms of preferring one probability distribution in comparison with another one.

Instead, people think in terms of gains and losses connected with definite states, and they do this exactly as one would have to use these notions if one would follow the decision-theoretic algorithm: So people distinguish between great and small gains and losses. They think about them not in absolute, but in relative terms. And these relative sizes of gains and losses become important for decision-making if uncertainty is involved: If one decides if a gain may be big enough, or, instead, too small in comparison with a loss, given the probability of the loss. Or if one decides which probability of a loss would be acceptable given the relation between the gain and the loss.

The second argument is the very simplicity of the data one needs to define the aims in cardinal utility: One needs a single function $u(x)$ on the set of definite states X. The comparatively simple notion of an order $\succ$ between definite states X is not viable for decision-making under uncertainty, so, is not acceptable. The viable concept of an order $\rhd$ on $\mathcal{P}(X)$ is, instead, much more complicate: It is a binary relation $\mathcal{P}(X) \times \mathcal{P}(X) \to \{0, 1\}$, which, moreover, has to fulfill a lot of additional restrictions – the various rationality principles. Instead, exactly the same standard of rationality is obtained by cardinal utility without any restrictions for the utility function $u(x)$.

Then there is also a philosophical argument, which becomes especially strong if one follows a Bayesian, subjectivist notion of probability: It is rational to define the aims in terms of real outcomes, not in terms of subjective beliefs. This remains valid even if one has to make the decisions based on subjective beliefs about their plausibility: The subjective beliefs are only a tool to make reasonable choices. But the aims themself should be defined in terms of the really possible outcomes. This argument becomes less impressive in the frequentist interpretation, but remains an interesting argument even in this case: Even if a probability is in some sense objective, we never really, actually appear in a probabilistic "state".

Nonetheless it seems important to note that the main points of this paper do *not* depend on the acceptance of the arguments of this section. Given the mathematical equivalence of rational probabilistic ordinal utilities $\rhd$ and cardinal utilities $u(x)$, ordinal utility remains viable in principle, as a philosophical concept, and the cardinal utility $u(x)$ can be interpreted as a particular method to define the ordinal utility $\rhd_u$.

The standard argumentation against cardinal utility appears nonetheless invalid and has to be rejected.

15. Consequences of cardinal utility

What has to be changed in Austrian economics if one accepts cardinal utility as valid? This has to be left to future research, and in the following I present only a few premature ideas.

It seems reasonable to guess that all of the positive results of Austrian economics survive. In fact, what has been proven without using cardinal utility remains proven if cardinal utility is allowed. There remains a caveat: If the proof has used the hypothesis that ordinal utility is sufficient even in the case of uncertainty, one has to take a look if the correct notion of ordinal utility – the order $\rhd$ on $\mathcal{P}(X)$ instead of $\succ$ on X – has been used. But I'm not aware of a result of Austrian economics where this caveat applies.

On the other hand, the arguments of Austrians against the mainstream have to be reconsidered: Does it depend on the point that cardinal utility is meaningless or unobservable? Possibly, some of these arguments have to be rejected completely, other modified or weakened.

In particular some formulas rejected now by Austrians as completely meaningless become meaningful. There is, for example, the *approximate connection* between money and utility. If we denote with n€ the state where I own n €, then

$$\frac{u(n_2e) - u(n_0e)}{u(n_1e) - u(n_0e)} \approx \frac{n_2 - n_0}{n_1 - n_0} \tag{5}$$

It is only an approximation. Moreover, its domain of applicability is restricted to amounts $n_0 \sim n_1 \sim n_2 \not\sim 0$, which are sufficiently close to each other in their size, so that the decreasing utility for money remains unimportant. But the approximate character of such formulas as well as their restricted domain of applicability is usually not questioned even by the proponents. So, one may say that the argument that such formulas are meaningless is only weakened. But essentially it has to be given up.

Let's, for example, consider the continuation of our first quote from Rothbard:

> Value scales of each individual are purely ordinal, and there is no way whatever of measuring the distance between the rankings; indeed, any concept of such distance is a fallacious one. Consequently,

> there is no way of making interpersonal comparisons and measurements, and no basis for saying that one person subjectively benefits more than another. ([1] p.258)

This rigorous rejection of any interpersonal utility comparison cannot be preserved. Interpersonal utility comparison remains quite arbitrary and problematic, but is not completely meaningless.

On the other hand, I see the advantage of increasing internal consistency of the Austrian approach. The reason is that I consider cardinal utilities to be an important, even necessary part of the "verstehen" of human motivation for their actions. So I see a contradiction between the Misesian concept of "verstehen" and the rejection of the cardinal aspects of utility. In fact, I see the refusal to accept cardinal utility as an essentially behaviouristic idea. But this is not the place to discuss this in detail.

16. Conclusion

The main result of this paper is that an important restriction in Austrian economics – the argumentation against cardinal utility – has to be rejected as unjustified. Cardinal utilities matter for rational decision-making in an uncertain world in agreement with common sense. Ordinal utility in the simple, common sense meaning – as an order $x \succ x'$ between definite states $x, x' \in X$ – is not sufficient: I have presented a situation where people have an identical ordinal utility but nonetheless different interests which lead to different decisions. I have shown that – and there – the standard argument about unobservability of the cardinal aspects fails. In particular, I have shown that the affine structure of cardinal utility is observable in principle. Only a linear transformation $F(u) = au+b;\ a > 0$ remains unobservable.

Cardinal utility becomes important for decision-making under uncertainty, where the relative sizes of the possible losses and gains make a difference, and the preferences (winning better than not risking better than loosing) is not a sufficient base for the decision to risk. The argument does not depend on the question of interpretation of probability – it holds for subjective probability as well as for frequentist probability.

Ordinal utility remains valid as a theoretical concept, as an order $\rhd$ on the space of probability distributions. Then cardinal utility is only a particular algorithm which defined such an order $\rhd_u$ for a given utility function $u(x)$ defined on definite states. But orders $\rhd$ not obtained as $\rhd_u$ of some cardinal utility $u(x)$ can be rejected as irrational: I present a few self-evident rationality principles so that for orders $\rhd$ which fulfill them the cardinal utility function $u(x)$ can be reconstructed. So this generalized concept of ordinal utility does not give anything interesting.

I would like to mention that there are other arguments in favour of cardinal utility not considered here. In fact, a case can be made that cardinal utility is necessary already for rational decision-making in a predictable world, so that the results of this paper should not be misinterpreted as the thesis that cardinal utility matters only for decision-making under uncertainty. Then, a methodological case can be made that a shift from ordinal to cardinal utility is a further shift away from empiricism and behaviourism. These questions could be considered in detail in future papers.

Block's characterization of the rejection of cardinal utility as "one way to distinguish the Austrian School from the mainstream" suggests the importance of this point for Austrian economics. In fact, acceptance of the conclusions of this paper would mean that Austrian arguments against mainstream economics have to be reevaluated and weakened, and in part given up. So with an acceptance of cardinal utility praxeology automatically becomes more close to the mainstream, in particular to decision theory.

I think it would be an advantage for libertarian economic theory if its Austrian and non-Austrian parts become more close to each other.

Appendix A. Proof of the theorems

Let's prove now the theorems 2, 4, and 5.

Because of theorem 1 the rationality principles hold for the assumptions of theorem 2, and, therefore, we can use them in the proof of all theorems.

Given an order, one can always define a corresponding equivalence relation, a notion of indifference, which I will denote for the order $\rhd$ by $\triangleq$. It is defined by

$$d\mu \triangleq d\mu' \quad \Leftrightarrow \quad (d\mu \trianglerighteq d\mu') \wedge (d\mu \trianglelefteq d\mu'). \tag{6}$$

Combining the principles of preference for better outcome for $\trianglerighteq$ and $\trianglelefteq$, we obtain a *principle of indifference* regarding indifferent outcomes:

$$d\mu \triangleq d\mu' \quad \Rightarrow \quad (pd\mu + (1-p)d\omega) \triangleq (pd\mu' + (1-p)d\omega). \tag{7}$$

Let's consider now two different definite states $\hat{x}_0 \lhd \hat{x}_1$. Then we can consider the interval between them consisting of the probability distributions

$$d\omega_p = p\hat{x}_1 + (1-p)\hat{x}_0; \qquad 0 \le p \le 1 \tag{8}$$

The principle of preference for better chances defines the order on this line completely: $d\omega_p \trianglerighteq d\omega_q$ if $p \ge q$.

Let's construct now the gain $r(x)$. Assume at first that the third state $\hat{x}$ is between the reference states: $\hat{x}_0 \trianglelefteq \hat{x} \trianglelefteq \hat{x}_1$. Then there exists a single value of indifference r defined by $d\omega_r \triangleq \hat{x}$. This value of indifference can be found by the standard Dirichlet interval subdivision: Once we have found an interval $[a, b]$ with $d\omega_a \trianglelefteq \hat{x} \trianglelefteq d\omega_b$, we compare $\hat{x}$ with $d\omega_{\frac{a+b}{2}}$. Whatever the result, at least one of the smaller intervals $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$ contains $\hat{x}$. Starting with $[0, 1]$, we can construct in this way an infinite sequence of intervals $[a_i, b_i]$ so that $d\omega_{a_i} \trianglelefteq \hat{x} \trianglelefteq d\omega_{b_i}$ with decreasing length $|b_i - a_i| = 2^{-i} \to 0$. So their limits define a single real number r by $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = r$. The principle of continuity gives now $d\omega_r = \lim_{i\to\infty} d\omega_{a_i} \trianglelefteq \hat{x}$ and $d\omega_r = \lim_{i\to\infty} d\omega_{b_i} \trianglerighteq \hat{x}$, which combines to $d\omega_r \triangleq \hat{x}$.

This $r(x)$ is uniquely defined, because, if $d\omega_{r'} \triangleq x$, it follows from transitivity of equivalence that $d\omega_{r'} \triangleq d\omega_r$, which is possible only if $r' = r$ because of the principle of preference for the better chances.

We have also $r(x_0) = 0$, $r(x_1) = 1$ by construction.

Assume now that the order $\rhd$ is the result $\rhd_u$ of application of (1) to some deterministic cardinal utility $u(x)$. It follows that if $d\mu \triangleq_u d\mu'$ then $\langle u\rangle(d\mu) =$

$\langle u\rangle(d\mu')$. Now we have $\hat{x} \triangleq_u d\omega_r$. It follows that

$$\begin{aligned} u(x) &= \langle u\rangle(\hat{x}) && \text{definition of } \langle u\rangle, \hat{x} \\ &= \langle u\rangle(d\omega_r) && \hat{x} \triangleq_u d\omega_r \\ &= \langle u\rangle(r\hat{x}_1 + (1-r)\hat{x}_0) && \text{definition of } d\omega_r \\ &= r\langle u\rangle(\hat{x}_1) + (1-r)\langle u\rangle(\hat{x}_0) && \text{linearity of } \langle u\rangle \\ &= ru(x_1) + (1-r)u(x_0) && \text{definition of } \langle u\rangle, \hat{x}_0, \hat{x}_1 \\ &= r(u(x_1) - u(x_0)) + u(x_0). \end{aligned} \tag{9}$$

So we have proven (4) for this case.

This result remains valid also for $\hat{x} \lhd \hat{x}_0$ or $\hat{x} \rhd \hat{x}_1$. To prove this, it is sufficient to rename the states as $x' = x_0$, $x'_0 = x$, $x'_1 = x_1$ resp. $x' = x_1$, $x'_0 = x_0$, $x'_1 = x$ to fit into the conditions $\hat{x}'_0 \lhd \hat{x}' \lhd \hat{x}'_1$. Then, the same considerations show the same results for some $r'(x')$. Because we have strong inequalities here, we can exclude the cases $r' = 0$ and $r' = 1$. So we can divide by r' as well as by $(1-r')$ as necessary. Renaming back gives now the same formula (4) for $r(x) = -1/(1 - r'(x'))$ resp. $r(x) = 1/r'(x')$. So (4) holds for all x. So theorem 5 has been proven.

As a consequence, the deterministic cardinal utility u is uniquely defined by the probabilistic ordinal utility $\rhd$ and the two real numbers $u(x_1)$ and $u(x_0)$. These two numbers can be freely chosen, except for the condition $u(x_1) > u(x_0)$, which follows from $\hat{x}_1 \rhd \hat{x}_0$ and (1).

For two different choices $u(x_1) > u(x_0)$, $u'(x_1) > u'(x_0)$ the $a > 0$, $b \in \mathbb{R}$ so that $u'(x) = au(x) + b$ holds for $x \in \{x_0, x_1\}$ can be easily found:

$$a = \frac{u'(x_1) - u'(x_0)}{u(x_1) - u(x_0)} > 0; \qquad b = u'(x_0) - u(x_0) \tag{10}$$

Because all other $u(x)$, $u'(x)$ can be computed from the values for x_0, x_1 in a linear way by (4) with the same $r(x)$, the relation $u'(x) = au(x) + b$ holds for all other x too. So theorem 2 is proven too.

It remains to prove the existence. Here we restrict ourself to a finite set X. We use now the minimum as $\hat{x}_0$ and the maximum as $\hat{x}_1$ and have, then, by the same construction, for all $\hat{x}$ a value $r(x)$ with $\hat{x} \triangleq d\omega_{r(x)}$. Now we can simply use $u(x) = r(x)$.

Let's consider now linear combinations of definite states $d\mu = \sum p_i \hat{x}_i$. The indifference principle allows to replace every definite state $\hat{x}_i$ in such a combination by the corresponding $d\omega_{r(x_i)}$. So we can obtain

$$\begin{aligned} d\mu &= \sum p_i \hat{x}_i \\ &\triangleq \sum p_i d\omega_{r(x_i)} \\ &= \sum p_i (r(x_i)\hat{x}_1 + (1 - r(x_i))\hat{x}_0) \\ &= \sum p_i r(x_i)\hat{x}_1 + (1 - \sum p_i r(x_i))\hat{x}_0 \\ &= \langle r\rangle(d\mu)\hat{x}_1 + (1 - \langle r\rangle(d\mu))\hat{x}_0 \\ &= d\omega_{\langle r\rangle(d\mu)} \end{aligned} \tag{11}$$

So $d\mu \triangleq d\omega_{\langle r\rangle(d\mu)}$. Let's consider now two arbitrary states $d\mu \rhd d\mu'$. Because of (11) this is equivalent to $d\omega_{\langle r\rangle(d\mu)} \rhd d\omega_{\langle r\rangle(d\mu')}$, which is, because of the principle of preference for better chances, equivalent to $\langle r\rangle(d\mu) > \langle r\rangle(d\mu')$. Thus, we have constructed a deterministic utility function $r(x)$ which defines the order $\rhd$ by (1), thus, theorem 4 is proven too.

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